Solution by J. SCHEFFER, A. M., Hagerstown, Md.

This is the famous Riccati equation. It can be solved only under certain circumstances.

If $m=0$, we have $\dfrac{dy}{ay^2+b}+dx=0$, easily integrated. If $m=-2$, $\dfrac{dy}{dx}=ay^2+\dfrac{b}{x^2}$ becomes homogeneous, if y is replaced by z^{-1}. If $m=-4$, we put $y=\dfrac{1}{ax}+\dfrac{z}{x^2}$, then we have $\dfrac{dz}{az^2+b}+\dfrac{dx}{x^2}=0$, which can easily be integrated. When m has the form $-\dfrac{4n}{2n-1}$ or $-\dfrac{4n}{2n+1}$ the equation yields integrable forms, putting in the former case $x=1/u$, in the latter $y=1/z$.

This problem is fully treated in Johnson's *Differential Equations*, Chapter IX, and in Forsyth's *Differential Equations*, second edition, pages 170-176. A number of our contributors referred to these and other sources. Ed.F.

MECHANICS.

240. Proposed by S. A. COREY, Hiteman, Iowa.

A perfectly flexible wire rope weighing one pound per foot is suspended from the tops of two vertical supports 300 feet apart, one support being 30 feet higher than the other. One end of the rope is fastened to the top of the higher support, while 600 feet of the rope hangs vertically from the top of the lower support. Assuming that the rope is free to slide over the top of the lower support without friction, find the lowest point of that portion of the rope which is suspended between the supports. Also find the amount of work which must be performed in raising the lowest point to make it coincide with the top of the lower support by exerting a pull on the free end of the rope.

Discussion by F. H. SAFFORD, Ph. D., University of Pennsylvania.

This problem involves several points of theoretical interest which may be better treated as a general problem. The analysis will be carried to such a stage that the transcendental equations involved may be used to obtain numerical results by any convenient method of computation.

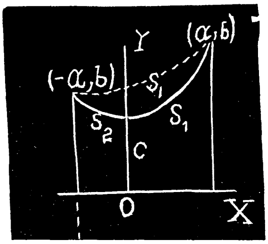
It is evident that between the supports the rope will hang in a catenary which, if referred to appropriate axes to be found, has the equation

$$y=c\cosh\frac{x}{c}. \tag{1}$$

Let the tops of the two supports be $(-a, b)$ and (a_1, b_1), and let the free end of the rope hang from the first point.

"The tention at any point of the catenary is equal to the weight of a portion of the string whose length is equal to the ordinate of the point" (Minchin's *Statics*, *or* Bowser's *Analytic Mechanics*). With this use of the term "ordinate" the X-axis must be the directrix of the catenary, and since in this problem the rope is of unit length, weights and lengths are numeric-

ally the same. Thus b is the length of the free end in the initial position of the rope. It is desirable to express the constants in the problem in terms of quantities which from the point of view of Mechanics would naturally be given, *i. e.*, the difference in level of the supports, the perpendicular distance between them and the length of the rope in the catenary. Thus with the notation used above let us write



$$\begin{aligned} a+a_1&=2d,\\ b_1-b&=h,\\ s+s_1&=l, \end{aligned} \tag{2}$$

where s and s_1 are the lengths of the left and right portions of the catenary measured from the vertex. Since the curve passes through the points $(-a, b)$ and (a_1, b_1) it follows from (1) that

$$\begin{aligned} b&=c\cosh\left(\frac{-a}{c}\right)=c\cosh\frac{a}{c}\\ b_1&=c\cosh\frac{a_1}{c}. \end{aligned} \tag{3}$$

The values of s and s_1 are found to be

$$s=c\sinh\frac{a}{c}, \quad s_1=c\sinh\frac{a_1}{c}. \tag{4}$$

From (2), (3), and (4),

$$\begin{aligned} h&=b_1-b=c\left[\cosh\frac{a_1}{c}-\cosh\frac{a}{c}\right]=2c\sinh\frac{a+a_1}{2c}\sinh\frac{a_1-a}{2c}\\ l&=s+s_1=c\left[\sinh\frac{a_1}{c}+\sinh\frac{a}{c}\right]=2c\sinh\frac{a+a_1}{2c}\cosh\frac{a_1-a}{2c}. \end{aligned} \tag{5}$$

$$\begin{aligned} h&=2c\sinh\frac{d}{c}\sinh\frac{a_1-a}{2c}\\ l&=2c\sinh\frac{d}{c}\cosh\frac{a_1-a}{2c}. \end{aligned} \tag{6}$$

$$\sqrt{(l^2-h^2)}=2c\sinh\frac{d}{c}, \tag{7}$$

$$a_1-a=2c\tanh^{-1}\frac{h}{l}. \tag{8}$$

Equation (7), though transcendental, is the most important result at this stage, as it may be shown that from it a single positive value of c is obtainable. From (8) and the first (2) followed by (3)

$$\left.\begin{aligned}
a&=d-c\tanh^{-1}\frac{h}{l},\\
a_1&=d+c\tanh^{-1}\frac{h}{l},\\
b&=c\cosh\left[\frac{d}{c}-\tanh^{-1}\frac{h}{l}\right],\\
b_1&=c\cosh\left[\frac{d}{c}+\tanh^{-1}\frac{h}{l}\right].
\end{aligned}\right\} \tag{9}$$

Thus from (7) and (9) the positions of the vertex and directrix of the catenary may be found. [In the original numerical problem $b=600$, $b_1=630$, $2d=300$, and the elimination of a and a_1 from (3) and the first of (2) gives

$$c\left[\cosh^{-1}\frac{b}{c}+\cosh^{-1}\frac{b_1}{c}\right]=2d. \tag{10}$$

The value of c must be found from (10) and then a and a_1 from (3), after which l follows from (7) and the solution may proceed as in the general case.]

The work performed in raising the lowest point to make it coincide with the top of the lower support by exerting a pull on the free end of the rope may be computed from the difference in level of the center of gravity of the entire rope before and after the change in position. Assuming that the rope will admit of pushing as well as pulling, the work may even be negative. In taking a and a_1 positive the lowest point is between the supports, but when a is 0 the vertex is at the top of the lowest support and there is no work to be done. When a is not 0 the problem is possible provided s_1 is less than the distance between the tops of the supports, *i. e.*,

$$c\sinh\frac{a_1}{c} < \sqrt{(h^2+4d^2)}. \tag{11}$$

This inequality is the closest criterion to be imposed upon the given quantities l, h, and d, others arising earlier being omitted because now superfluous. The ordinate of the center of gravity of the rope in the two positions involves the determination, for the catenary, of

$$\int y ds = c\int \cosh^2\frac{x}{c}dx = \frac{c^2}{4}\sinh\frac{2x}{c} + \frac{cx}{2}. \tag{12}$$

Between the limits $-a$ and a_1, (12) gives

$$\int_{-a}^{a_1} y ds = \frac{c^2}{2}\sinh\frac{2d}{c}\cosh 2\left(\tan^{-1}\frac{h}{l}\right) + cd$$

$$= \frac{c^2}{2}\cdot\frac{l^2+h^2}{l^2-h^2}\sinh\frac{2d}{c} + cd. \tag{13}$$

Taking into account the free end of the rope, the center of gravity of the entire rope in the initial position may now be found.

For the final state of the rope a new c must be computed in terms of h, d, and a new l which is the original s_1, [vide (4)], $\bar{c}$ being given by

$$\sqrt{(s_1^2-h^2)} = 2\bar{c}\sinh\frac{d}{\bar{c}}. \tag{7a}$$

The equation of the curve is now

$$y = \bar{c}\cosh\frac{x}{\bar{c}}, \tag{1a}$$

(referred to new axes), and the positions of the new vertex and directrix may be computed as before. The limits in (12) are to be taken as 0 and $2d$, c becoming $\bar{c}$, hence

$$\int_0^{2d} y ds = \frac{\bar{c}^2}{4}\sinh\frac{4d}{\bar{c}} + \bar{c}d. \tag{13a}$$

The free end is now longer by s, and the ordinate of the center of gravity of the new system may be found as before, though measured from the new directrix. The difference in level of the two centers of gravity may now be

readily obtained. In numerical problems the task of solving the transcendental equations (7) or (10), and (7a) presents no practical difficulty.

NUMBER THEORY AND DIOPHANTINE ANALYSIS.

172. Proposed by H. C. FEEMSTER, York, Neb.

Show that $\frac{(nr)!}{n!(r!)^n}$ is an integer.

Solution by G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa.

By induction we get the following:
Suppose $r(n-1)!$ is divisible by $(r!)^{n-1}(n-1)!$. Now

$$\frac{(rn)!}{n!(r!)^n} \div \frac{r(n-1)!}{(r!)^{n-1}(n-1!} = \frac{(nr)!}{r\ (n-1)!} \times \frac{(n-1)\ !}{n!r!}$$

$$= \frac{nr(nr-1)(nr-2)\ldots\text{to } r \text{ factors}}{nr(r-1)!}$$

$$= \frac{(nr-1)(nr-2)\ldots\text{to } (r-1) \text{ factors}}{(r-1)!} = \text{an integer.}$$

Now $\frac{r!}{r!\,1!}$=an integer, and hence $\frac{(2r)!}{(r!)^2 2!}$=an integer; $\frac{(3r)!}{(r!)^3 3!}$=an integer, and so on.

173. Proposed by V. M. SPUNAR, M. and E. E., East Pittsburg, Pa.

Find integral values satisfying the equation,

$$a_1^2 + a_2^2 + a_3^2 + \ldots + a_n^2 = d^4.$$

I. Solution by FRANK LOXLEY GRIFFIN, Ph. D., Williams College.

One set of solutions may be obtained by putting $a_1 = df_1$, $a_2 = df_2$, etc., which reduces the problem to that of finding a sequence of n integers, the sum of whose squares is a perfect square. Or, geometrically, we seek $n-1$ right triangles whose sides are all integers, and the hypotenuse of each being one leg of the next.

I. This is readily accomplished for $n=2$ by recalling that, if (p^2-q^2) and $2pq$ are legs, where p and q are integers, the hypotenuse is also an integer, (p^2+q^2). Let f_1 be *any odd integer* (>1) and take $p-q=1$ and $p+q=f_1$, so that $p=\frac{1}{2}(f_1+1)$, $q=\frac{1}{2}(f_1-1)$, both integers. Thus the sides